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Wave energy-momentum and pseudoenergy-momentum conservation for the layered quasi-geostrophic instability problem

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Evolution equations and conservation laws are derived for a quite general layered quasi-geostrophic model: with arbitrary thickness and stratification structure and with either a free or a rigid (including the possibility of topography) boundary condition, at the top and bottom. The system is shown to be Hamiltonian, and Arnol'd stability conditions are derived, in the sense of both the first and second theorem, i.e. for pseudowestward and pseudoeastward basic flows, respectively, and for arbitrary perturbations of potential vorticity and Kelvin circulations.

Two examples of parallel basic flow in a channel are analysed: the sine profile in the so-called *equivalent barotropic model* (one layer with a free boundary) and Phillips' problem (uniform flow in each of two layers with rigid boundaries). Using the second theorem with the optimum combination of pseudoenergy and pseudomomentum it is shown that, in both cases, the basic state is nonlinearly stable if the channel width L is small enough, namely, $AL < \pi$ and $2(f_0 L/\pi)^2 < g'(H_1 H_2)^{\frac{1}{2}}$, respectively. (In the first problem, A is the wavenumber of the sine profile; in the second one, g' is the reduced gravity, H_1 and H_2 are the layer thicknesses, and f_0 is the Coriolis parameter). The stability condition of either problem is found to be also a necessary one: as soon as it is violated a grave mode becomes unstable. It is shown explicitly that the second variation of the pseudoenergy and pseudomomentum of a growing (decaying) normal mode is identically zero, defining the direction of the unstable (stable) manifold.

1. Introduction

The method of Arnol'd (1965, 1966) is used to derive sufficient stability conditions of a hydrodynamical problem on the basis of its conservation laws. Although not indispensable for such a purpose, the Hamiltonian structure of the fully nonlinear system, if established, is very useful because it clearly spells out the relationship between symmetries and integrals of motion (Shepherd 1990). Two types of stability are usually studied: formal – which refers to the existence of an integral of motion $\Delta\mathscr{S}$ with a local extremum in phase space – and nonlinear – related to the existence of a *norm* of the perturbation, proved to be bounded by a multiple of its initial value (Holm *et al.* 1985; McIntyre & Shepherd 1987).

The orthodox application of Arnol'd's method can only be done for steady basic states of quasi-geostrophic models (QGM); in this case, $\Delta\mathscr{S}$ is the pseudoenergy $\Delta(\mathscr{E}+\mathscr{C}_0)$; if the basic state is also parallel, then the most useful Lyapunov functional is an arbitrary combination of the pseudoenergy and the pseudomomentum

$\Delta(\mathcal{M}+\mathcal{C}_1)$, say, $\Delta\mathcal{S} := \Delta(\mathcal{E}-\alpha\mathcal{M}+\mathcal{C}_0-\alpha\mathcal{C}_1)$. Arnol'd's first (second) theorem corresponds to $\mathcal{S}$ being a minimum (maximum) at the basic state, i.e. $\delta\mathcal{S} \equiv 0$ and $\delta^2\mathcal{S} > 0$ ($\delta^2\mathcal{S} < 0$). (The notation used is standard – but nevertheless reviewed below: e.g. Δ, δ and δ^2 denote the total, first and second variations of a functional from its value at the basic state, respectively.)

Both the Hamiltonian formalism and Arnol'd stability conditions (formal and nonlinear, first and second theorem, and using both pseudoenergy and pseudomomentum) have been worked out for the continuously stratified QGM (Holm 1986; Swaters 1986; McIntyre & Shepherd 1987; Shepherd 1989); in all these papers, the top and bottom boundary are rigid, and might be non-isopycnal (except for the published proof of Arnol'd's second theorem). In contrast, for a QGM with several homogeneous *layers*, only Arnol'd's first theorem has been derived (for the case of equal thicknesses and buoyancy jumps across interfaces, and rigid boundaries), using the Hamiltonian structure of the system (Holm *et al.* 1985). Benzi *et al.* (1982), on the other hand, derived the first and second theorem for the so-called equivalent barotropic QGM, which is no more than a one-layer system with a free boundary (top and/or bottom), in an infinite domain.

The first goal of this paper is to complete the treatment of the case with stepped density stratification. Section 2 presents the Hamiltonian structure and conservation laws of a quite general layered QGM, with arbitrary buoyancy jumps and layer thicknesses, and for both rigid (like Holm *et al.* 1985) and free (like Benzi *et al.* 1982) top and bottom boundary conditions (the first possibility might include topography). Arnol'd first and second theorems for this model are derived in §3.

Sufficient stability conditions are, of course, equivalent to necessary instability criteria. An unstable basic state must violate all possible stability conditions, e.g. in the case of a steady and parallel basic flow, $\delta^2(\mathcal{E}-\alpha\mathcal{M}+\mathcal{C}_0-\alpha\mathcal{C}_1)(\equiv\delta^2\mathcal{S})$ must be sign indefinite for any α. Moreover, $\Delta\mathcal{S} = 0$ is the way of escape from the basic state in phase space, i.e. $\delta^2\mathcal{S} = 0$ indicates the original direction of the unstable and the stable manifolds. Two particular examples of unstable flows are discussed: the sine profile in the equivalent barotropic model (one layer with free surface) in §4, and Phillips' problem (two layers with rigid top and bottom boundaries) in §5.

The general conclusions of this work are compared, in §6, with those of primitive equations models (such as shallow-water equations) and there is a discussion of which results are common and which are peculiar, and why. Finally, mathematical details of the proof of Arnol'd's second theorem and of the normal modes of Phillips' problem are left for Appendices A and B, respectively.

2. Model equations and conservation laws

In this section, I will first describe a quite general layered quasi-geostrophic system, including the possibility of an arbitrarily shaped domain, the rigid or free top and bottom boundaries. (I am using the Boussinesq approximation, but the equations can be easily changed for their application to the atmosphere.) Next, I will present the Hamiltonian and Poisson bracket which give the evolution equations. Finally, I will derive the form of the momenta, by making them the generators of the spatial transformations. For a comparison of layered and 'level' models, see Pedlosky (1979, §6.18).

Consider a system with N homogeneous layers, whose thicknesses at rest (not necessarily equal) are H_j, with $j = 1, \ldots, N$, from top to bottom. In each layer a streamfunction $\psi_j(x, y, t)$ is defined, which determines not only the velocity components along the x (eastward) and y (northward) directions by

$$u_j = -\partial_y \psi_j, \quad v_j = \partial_x \psi_j \quad (j = 1, \dots, N), \tag{2.1}$$

but also the vertical displacement ζ_j of the interface between the jth and $(j+1)$th layers, by the hydrostatic balance

$$g'_j \zeta_j = f_0(\psi_{j+1} - \psi_j) \quad (j = 1, \dots, N-1), \tag{2.2}$$

where g'_j is the buoyancy jump across that interface and the Coriolis parameter equals $f_0 + \beta y$.

The potential vorticity in a generic layer is

$$q_j = \partial_x v_j - \partial_y u_j - \frac{f_0}{H_j}(\zeta_{j-1} - \zeta_j) + \beta y, \tag{2.3}$$

$(j = 1, \dots, N)$. Evaluation of the potential vorticity fields in the first and last layers requires knowledge of ζ_0 and ζ_N; these are determined by the 'horizontal' boundary conditions, discussed next. Two choices are possible for each one, a rigid or soft boundary. In the first case the corresponding ζ is a given function of horizontal position, $\tau(\boldsymbol{x})$, which models topographic effects (including, of course, the possibility $\tau \equiv 0$), whereas in the second one the boundary is free. Namely

$$\zeta_0 = \tau_0(x, y) \quad (a := 1), \tag{2.4a}$$

or

$$g'_0 \zeta_0 = f_0 \psi_1(x, y, t) \quad (a := 0), \tag{2.4b}$$

for the top boundary, and

$$\zeta_N = \tau_N(x, y) \quad (b = N-1), \tag{2.5a}$$

or

$$g'_N \zeta_N = -f_0 \psi_N(x, y, t) \quad (b := N), \tag{2.5b}$$

for the bottom one; the parameters a and b are here defined for future use.

For example, the system studied by Benzi *et al.* (1982) corresponds to $N = 1$, $\tau_1 \equiv 0$ and the choices (2.4*b*) and (2.5*a*), whereas that discussed in Holm *et al.* (1985) has both boundaries rigid, (2.4*a*) and (2.5*a*), and the g'_j and H_j independent of j (otherwise, their equation (4A1) is incorrect; this corresponds to a uniform discretization of the case of constant Brunt–Väisälä profile). The free boundary conditions (2.4*b*) and (2.5*b*) may be formally obtained by making $\psi_0 = 0$ and $\psi_{N+1} = 0$ in (2.2). However, there is no need for the assumption of passive, motionless, layers; e.g. the model in Benzi *et al.* (1982) may well represent a one-layer system, with a free surface and vacuum above it.

The Kelvin circulations are defined by $\gamma_j^\nu := \oint \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{u}_j$, with the integral made along $\partial\mathfrak{D}_\nu$, which denotes each connected part of the boundary of the horizontal domain $\mathfrak{D}$ (e.g. each wall, in the case of a channel). It is possible to show that the state of the system is fully and uniquely determined, at any time, by $q_j(\boldsymbol{x}), \gamma_j^\nu$ and the no-flow condition in each solid boundary, $\boldsymbol{u}_j \cdot \boldsymbol{n} \equiv 0, \boldsymbol{x} \in \partial\mathfrak{D}$, where $\boldsymbol{n}$ is the outward unit vector: the $\psi_j(\boldsymbol{x})$ can be inverted (in some cases, up to the addition of an irrelevant constant) from that information. To be precise, the volume integral of q equals a linear combination of the γ_j^ν, and thus these variables are not strictly independent; this point will be considered further in the following section; meanwhile, state variables are taken as q and γ.

The Hamiltonian structure of the problem is evidenced by the existence of a functional of state $\mathscr{H}$ and an appropriate Poisson bracket $\{.,.\}$ such that the time derivative of any functional of state, $\mathscr{A}[q,\gamma] = \int d^2x A(..,x,t)$, is given by

$$\frac{d\mathscr{A}}{dt} = \{\mathscr{A},\mathscr{H}\} + \int_{\mathfrak{D}} d^2x \frac{\partial A}{\partial t}. \tag{2.6}$$

The Hamiltonian and Poisson bracket for this system are

$$\mathscr{H}[q,\gamma] := \frac{1}{2}\int_{\mathfrak{D}} d^2x \left(\sum_{j=1}^{N} H_j(u_j^2+v_j^2) + \sum_{j=a}^{b} g'_j \zeta_j^2 \right), \tag{2.7}$$

where a and b are defined in (2.4) and (2.5), and

$$\{\mathscr{A},\mathscr{B}\} := \int_{\mathfrak{D}} d^2x \sum_{j=1}^{N} H_j q_j \mathfrak{J}\left(\frac{\delta\mathscr{A}}{\delta q_j}, \frac{\delta\mathscr{B}}{\delta q_j}\right), \tag{2.8}$$

where $\mathfrak{J}(a,b) := a_x b_y - a_y b_x$ is the Jacobian, and the functional derivatives of any state functional are defined by

$$\mathscr{A}[q+\delta q, \gamma+\delta\gamma] - \mathscr{A}[q,\gamma] = \sum_{j=1}^{N} H_j \left(\int_{\mathfrak{D}} d^2x \frac{\delta\mathscr{A}}{\delta q_j}\delta q_j + \sum_{\nu} \frac{\delta\mathscr{A}}{\delta\gamma_j} d\gamma_j^\nu \right) + O(\delta q, \delta\gamma)^2. \tag{2.9}$$

The Poisson bracket (2.8) does have the required properties, among which the harder to prove (but a crucial one) is the Jacobi identity, namely

$$\{\mathscr{U},\{\mathscr{V},\mathscr{W}\}\} + \{\mathscr{V},\{\mathscr{W},\mathscr{U}\}\} + \{\mathscr{W},\{\mathscr{U},\mathscr{V}\}\} = 0, \tag{2.10}$$

for any three admissible functionals of state $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$. McIntyre & Shepherd (1987) give a detailed derivation of the Jacobi identity for the one-layer problem, appropriate for readers – like myself – who have not mastered hard core mathematics; that derivation yields to a careful definition of an admissible functional of state and is immediately generalizable to (2.8).

The evolution equations, derived from (2.6), are the following: in the first place there is constancy of Kelvin circulations, $d\gamma_j^\nu/dt = 0$, since $\{\gamma_j^\nu, \mathscr{H}\} \equiv 0$ because δ/δ_γ does not enter in the definition of the Poisson bracket (2.8). Secondly, the first variation of $\mathscr{H}$ is

$$\delta\mathscr{H} = \int_{\mathfrak{D}} d^2x \sum_{j=1}^{N} H_j \nabla\psi_j \cdot \nabla\delta\psi_j + \sum_{j=a}^{b} f_0(\psi_{j+1} - \psi_j)\,\delta\zeta_j;$$

integrating by parts the first term and rearranging the second sum (which amounts to an integration by parts in the vertical), $\delta\mathscr{H}/\delta q_j \equiv -\psi_j$. On the other hand, it is

$$\frac{\delta q_i(\boldsymbol{x}')}{\delta q_j(\boldsymbol{x})} = \delta(x-x')\,\delta(y-y')\,\delta_{ij} H_j^{-1}, \tag{2.11}$$

and consequently (2.6) gives

$$\partial_t q_i = \{q_i, \mathscr{H}\} \equiv \mathfrak{J}(q_i, \psi_i), \tag{2.12}$$

which is the law of potential vorticity conservation.

Equations (2.1)–(2.8) provide the generalization of those of Holm *et al.* (1985), for systems with non-uniform g' and H profiles and/or with free horizontal boundaries.†

† The formulae in Holm *et al.* (1985) are not free from typographical errors, since their $\mathscr{H}$ and $\{.,.\}$ do not yield the correct evolution equations.

Notice that $\sum_j H_j \ldots$ in (2.8) and $\delta_{ij} H_j^{-1}$ in (2.11) have the correct limits, $\int \mathrm{d}z \ldots$ and $\delta(z-z')$, respectively, when the vertically continuous case (with isopycnal horizontal boundaries) is obtained by making $N \rightarrow \infty$, $H_j \rightarrow 0$. (One could, if so desired, dispense with the H factors in both (2.9) and (2.11) changing H into H^{-1} in (2.8).)

A Casimir $\mathscr{C}$ is a functional of state $\mathscr{C}$ such that

$$\{\mathscr{A}, \mathscr{C}\} \equiv 0, \tag{2.13}$$

for any admissible functional of state $\mathscr{A}$ (McIntyre & Shepherd 1987). The Poisson bracket (2.8) suggests the form

$$\mathscr{C}[q,\gamma] = \sum_{j=1}^{N}\left(\int_{\mathfrak{D}} \mathrm{d}^2x H_j F_j(q_j) + \sum_{\nu} a_\nu \gamma_j^\nu\right), \tag{2.14}$$

where the functions F_j and the constants a_ν are arbitrary. Even though more complicated expressions can be written, this is general enough for all the applications in this paper.

The zonal momentum functional $\mathscr{M}^x$ is obtained by making it the generator of the transformation $q_j(x,y,t) \rightarrow q_j(x+\delta x, y, t)$, in the sense

$$\delta_x \mathscr{A} := \delta x \int_{\mathfrak{D}} \mathrm{d}^2x \sum_{j=1}^{N} H_j \frac{\delta A}{\delta q_j}(\partial_x q_j) \equiv \delta x\{\mathscr{M}^x, \mathscr{A}\}, \tag{2.15}$$

which demands $\delta\mathscr{M}^x/\delta q_j = y$ and an x-symmetric horizontal boundary (so that $-q(\delta\mathscr{A}/\delta q)_x$ may be replaced by $q_x(\delta\mathscr{A}/\delta q)$). From this equation and (2.11) it is easy to derive $\partial_x q_j = \{\mathscr{M}^x, q_j\}$, but this local transformation does not seem to be enough to determine $\mathscr{M}^x$ because it does not constrain the shape of $\partial\mathfrak{D}$.

Meridional and angular momenta $\mathscr{M}^y$ and $\mathscr{M}^a$ are similarly defined, as the generators of infinitesimal translations in y and rotations in the (x,y)-plane, respectively; their existence requires $\partial\mathfrak{D}$ to be invariant under the corresponding spatial transformation. More generally, a function $\mathscr{G}$ (conserved or not) is said to be the generator of the infinitesimal transformation $\delta_g \mathscr{A} := \delta g\{\mathscr{G}, \mathscr{A}\} + O(\delta g^2), \forall \mathscr{A}$; the Hamiltonian is, for instance, the generator of infinitesimal time translations (2.6), whereas the Casimirs are the generator of no transformation (2.13).

Using $\delta\mathscr{M}^x/\delta q_j = y$ in (2.9), choosing $\delta\mathscr{M}^x/\delta\gamma$ appropriately, and assuming an x-symmetric boundary, the explicit form

$$\mathscr{M}^x[q,\gamma] := \int_{\mathfrak{D}} \mathrm{d}^2x\left(\sum_{j=1}^{N} H_j u_j + f_0 y(\zeta_N - \zeta_0)\right) \tag{2.16}$$

is obtained. Notice that the contribution of the Coriolis potential (proportional to f_0) only matters for those cases in which a horizontal boundary is free, option (2.4*b*) and/or (2.5*b*), otherwise that term is trivially time-independent. Unlike the case of the primitive equations models (PEM), $\mathscr{M}^x$ is but linear in the state variables; consequently $\delta^n\mathscr{M}^x \equiv 0$ for $n > 1$.

Existence and conservation of $\mathscr{M}^x$ are not equivalent properties: the first (second) one is related to invariance of the boundary (the Hamiltonian) under x-translations; indeed

$$\{\mathscr{M}^x, \mathscr{H}\} \equiv \int_{\mathfrak{D}} \mathrm{d}^2x f_0(\psi_1 \partial_x \tau_0(\boldsymbol{x}) - \psi_N \partial_x \tau_N(\boldsymbol{x})), \tag{2.17}$$

where the first (second) term on the right-hand side is absent if the top (bottom) boundary is free, i.e. if (2.4*b*) [(2.5*b*)] is chosen, instead of (2.4*a*) [(2.5*a*)].

Therefore, if any of the horizontal boundaries is rigid and the corresponding topography τ is x-dependent, $\partial_x \tau \neq 0$, then $\mathscr{M}^x$ is not conserved, $\delta_t \mathscr{M}^x \not\equiv 0$, or, equivalently, $\mathscr{H}$ is not invariant under x-translations, $\delta_x \mathscr{H} \not\equiv 0$. In (2.7), $\mathscr{H}$ seems to be x-independent, because the right-hand side is written explicitly in terms of (u, v, ζ), i.e. in terms of the streamfunction, instead of q and γ: in the case of a non-symmetric topography, the calculation of the field ψ from (q, γ) introduces an implicit x-dependence in the Hamiltonian.

If each horizontal boundary is either free or has an x-independent topography, then $\{\mathscr{M}^x, \mathscr{H}\} \equiv 0$, i.e. $\mathscr{M}^x$ is conserved and $\mathscr{H}$ is x-independent. This implies also a symmetry of the problem, namely, making an infinitesimal x-translation of the initial condition and then letting the system evolve is equivalent to performing both operations in reverse order. For an admissible functional of state, the difference between both infinitesimal operations is

$$\delta_x(\delta_t \mathscr{A}) - \delta_t(\delta_x \mathscr{A}) = \delta x\, \delta t \{\mathscr{M}^x, \{\mathscr{A}, \mathscr{H}\}\} - \delta x\, \delta t \{\{\mathscr{M}^x, \mathscr{A}\}, \mathscr{H}\};$$

the right-hand side is equal to $\delta x\, \delta t \{\{\mathscr{H}, \mathscr{M}^x\}, \mathscr{A}\}$, by virtue of the Jacobi identity (2.10), and therefore vanishes, in the case under consideration.

An equation similar to (2.17) for $\mathscr{M}^y$ or $\mathscr{M}^a$ involves also a contribution from the non-symmetric β term. Recall that existence of $\mathscr{M}^y$ or $\mathscr{M}^a$ requires $\partial\mathfrak{D}$ to have the corresponding symmetry; for the conservation of either one it is also necessary not only that any topography be symmetric, but also that $\beta \equiv 0$.

In sum, the integrals of motion at our disposal are the energy $\mathscr{E}$ (which coincides with the Hamiltonian $\mathscr{H}$), the Casimirs $\mathscr{C}$ and, if the boundary allows, the appropriate momenta $\mathscr{M}$. A Casimir may sometimes be added to $\mathscr{H}$ and/or $\mathscr{M}$ in order to build an integral of motion (pseudoenergy and/or pseudomomentum) which is quadratic to lowest order in the deviation from a given state, say, $(q, \gamma) = (Q, \Gamma)$. Namely, defining the first variation $\delta\mathscr{F}$, the second one $\delta^2\mathscr{F}$, etc., in terms of the total variation $\Delta\mathscr{F}$, for any functional of state $\mathscr{F}[q, \gamma]$ in the form

$$\Delta\mathscr{F} := \mathscr{F}[Q + \delta q, \Gamma + \delta\gamma] - \mathscr{F}[Q, \Gamma] := \delta\mathscr{F} + \tfrac{1}{2}\delta^2\mathscr{F} + \tfrac{1}{6}\delta^3\mathscr{F} + \dots, \tag{2.18}$$

with
$$\delta^n \mathscr{F} = O(\delta q, \delta\gamma)^n,$$

the quest is, given the basic state (Q, Γ), to find $\mathscr{F}$ such that $\delta\mathscr{F} \equiv 0, \forall(\delta q, \delta\gamma)$.

For instance, in the β-plane the total enstrophy (variance of potential vorticity) is proportional to the pseudomomentum relative to the resting ocean (Ripa 1981*a*), e.g. in the equivalent barotropic case without topography (Benzi *et al.* 1982) its density is $\frac{1}{2}(q - \beta y)^2 = \frac{1}{2}q^2 - \beta M^x - \nabla\cdot(\beta y \nabla\psi) - \frac{1}{2}\beta^2 y^2$, i.e. $-\beta$ times the pseudomomentum, plus some trivially constant terms. In the f-plane, on the other hand, the enstrophy of any layer which is conserved is but a particular Casimir, namely, $F_j(q) = \frac{1}{2}q^2\delta_{js}$ in (2.14) for the sth layer. The limit $\beta \to 0$ is singular, in the sense that the connection between (Eulerian) integrals of motion and (Lagrangian) symmetries is partially lost (Ripa 1981*b*).

3. Nonlinear stability and instability conditions

In this section I will derive sufficient stability (or necessary instability) conditions from the integrals of motion derived in the previous section. They apply to perturbations with an arbitrary structure. The cases of non-parallel and symmetric flows are treated separately. In both cases the law of pseudoenergy conservation is used, but in the second one the pseudomomentum is also available, raising the possibility of more powerful conditions.

If the state of the system is split, at any time, in the form $q = Q + \delta q$ and $\gamma = \Gamma + \delta\gamma$ (where uppercase symbols denote variables in a (given) basic state), the quest is to find conditions on (Q, Γ) that prevent the growth of some measure of the perturbation $(\delta q, \delta\gamma)$. Two definitions of stability will be used here (Holm *et al.* 1985; McIntyre & Shepherd 1987). Formal stability is based on the existence of a Lyapunov functional $\mathscr{S}[q,\gamma]$ which is an integral of motion, $\mathscr{S}[q,\gamma]_{t>0} =$ constant $\forall (q,\gamma)_{t=0}$, and is an isolated minimum (or maximum) at this state, $\delta\mathscr{S} \equiv 0$ and $\delta^2\mathscr{S} > 0$ (or $\delta^2\mathscr{S} < 0$) $\forall(\delta q, \delta\gamma)$; recall (2.18). Nonlinear stability, on the other hand, means that there exists a norm of the perturbation $\|(\delta q, \delta\gamma)\|$ which is bounded by a multiple of its original value. This, for instance, can be proved when the total variation of $\mathscr{S}$ satisfies $a\,\|(\delta q, \delta\gamma)\|^2 \leqslant |\Delta\mathscr{S}| \leqslant A\|(\delta q, \delta\gamma)\|^2$, for some positive constants a and A: constancy of $\Delta\mathscr{S}$ implies $\|(\delta q, \delta\gamma)\|_{t>0} \leqslant (A/a)^{\frac{1}{2}}\,\|(\delta q, \delta\gamma)\|_{t=0}$. Nonlinear stability implies formal stability, which in turn implies normal modes stability; therefore, normal modes instability implies formal instability, which in turn implies normed instability.

The set of basic states whose stability can be studied by this method is quite restricted. In the first place, constancy of $\mathscr{S}$ implies that it may be constructed from the energy $\mathscr{E}$, all Casimirs $\mathscr{C}$, and any conserved momentum $\mathscr{M}$. Secondly, the condition $\delta\mathscr{S} = 0$ implies that $\{\mathscr{S}, \mathscr{A}\} \equiv 0$ at $(q,\gamma) = (Q,\Gamma)\,\forall\mathscr{A}[q,\gamma]$, i.e. (Q,Γ) is invariant under the infinitesimal transformation generated by $\mathscr{S}$. Consequently, the basic state (Q,Γ) must be either steady, $\mathscr{S} = \mathscr{E} + \mathscr{C}_0$, or symmetric, $\mathscr{S} = \mathscr{M} + \mathscr{C}_1$.

3.1. *Non-parallel steady flow*

If the basic state is steady, $\partial_t \Psi_j = 0$, (2.12) implies $\mathfrak{J}(\Psi, Q) = 0$, in each layer, i.e. $\Psi_j = \Psi_j(Q_j)$ (for a discussion of multivalued functions $\Psi(Q)$, see McIntyre & Shepherd 1987) and there is, at most, one Casimir $\mathscr{C}_0$ such that $\delta\mathscr{S} = 0$, with $\mathscr{S} = \mathscr{E} + \mathscr{C}_0$ (recall that $\mathscr{E} \equiv \mathscr{H}$). The Lyapunov functional is, then, the pseudoenergy

$$\Delta\mathscr{S} = (\Delta - \delta)\,\mathscr{E} + (\Delta - \delta)\,\mathscr{C}_0 = \text{constant}. \tag{3.1}$$

The energy part of $\Delta\mathscr{S}$ takes the form

$$(\Delta - \delta)\,\mathscr{E} \equiv \frac{1}{2}\int_{\mathfrak{D}} \mathrm{d}^2x \left(\sum_{j=1}^{N} H_j (\nabla\,\delta\psi_j)^2 + \sum_{j=a}^{\ell} g_j'(\delta\zeta_j)^2 \right) =: [\delta\psi, \delta\psi], \tag{3.2}$$

where a and ℓ are defined in (2.4) and (2.5); the inner product $[\delta\psi, \delta\psi]$ is here defined for future use. Notice that $(\Delta - \delta)\,\mathscr{E} \equiv \frac{1}{2}\delta^2\mathscr{E}$, i.e. $\delta^n\mathscr{E} \equiv 0\,\forall n > 2$; this is unlike the PEM case, for which $\delta^3\mathscr{E} \not\equiv 0$. Since $\delta\mathscr{H}/\delta q_j = -\Psi_j$ at $(q,\gamma) = (Q,\Gamma)$, the Casimir part of $\Delta\mathscr{S}$ takes the form

$$(\Delta - \delta)\,\mathscr{C}_0 \equiv \int_{\mathfrak{D}} \mathrm{d}^2x \sum_{j=1}^{N} H_j\,\sigma_j(Q_j, \delta q_j) = \langle \sigma \rangle, \tag{3.3}$$

with
$$\begin{aligned} \sigma_j(Q_j, \delta q_j) &:= \int_{Q_j}^{Q_j+\delta q_j} [\Psi_j(s) - \Psi_j(Q_j)]\,\mathrm{d}s \\ &\sim \tfrac{1}{2}\Psi_j'(Q_j)\,(\delta q_j)^2 + O(\delta q_j)^3, \end{aligned} \tag{3.4}$$

and $\langle\,..\,\rangle := \iint \mathrm{d}^2x \sum_j H_j$.

Unlike the case of primitive equations, $\delta^2\mathscr{E}$ is *a priori* positive definite; formal stability is then guaranteed if

$$\frac{\mathrm{d}\Psi_j}{\mathrm{d}Q_j} > 0, \tag{3.5}$$

which implies $\delta^2\mathscr{C}_0$ positive definite. In order to prove normed stability, ssume that there exist pairs of positive constants a_j and A_j such that

$$0 < a_j \leqslant \sigma_j(Q_j, \delta q_j)/\delta q_j^2 \leqslant A_j \quad (\forall \delta q_j \neq 0); \tag{3.6}$$

constancy of $\Delta\mathscr{S}$ implies $\langle \delta \boldsymbol{u}_j^2 + a_j \delta q_j^2 \rangle_{t>0} \leqslant \langle \delta \boldsymbol{u}_j^2 + A_j \delta q_j^2 \rangle_{t=0}$, and therefore defining $\|(\delta q, \delta\gamma)\| := \langle \delta \boldsymbol{u}_j^2 + a_j \delta q_j^2 \rangle^{\frac{1}{2}}$, it then follows that $\|(\delta q, \delta\gamma)\|_{t>0} \leqslant \mathrm{Max}\,(A_j/a_j)^{\frac{1}{2}} \|(\delta q, \delta\gamma)\|_{t=0}$.

Conditions (3.5) and (3.6) constitute Arnol'd's first theorem for the general layered QGM of the last section; they were first obtained by Benzi *et al.* (1982) for a one-layer case with a free surface and by Holm *et al.* (1985) for the multi-layer problem with rigid horizontal boundaries (and g_j' and H_j independent of j). They have the same form as in the continuously stratified case with isopycnal horizontal boundaries (Swaters 1986; McIntyre & Shepherd 1987), in sharp contrast with the PEM, for which the layered and continuously stratified cases have quite different *a priori* stability properties (Ripa 1990).

For the purposes of the following section, however, I need to derive Arnol'd's second theorem, which guarantees stability in cases where $\mathrm{d}\Psi/\mathrm{d}Q$ is negative everywhere, instead of (3.5). This might be possible in systems with free boundaries (Benzi *et al.* 1982) and/or for domains $\mathfrak{D}$ that are small enough in at least one dimension (McIntyre & Shepherd 1987) because, since $\delta\psi$ is a (non-local) function of $(\delta q, \delta\gamma)$, the wave energy $\delta^2\mathscr{E}$ may be bounded by $|\delta^2\mathscr{C}_0| (= -\delta^2\mathscr{C}_0)$ so that the pseudoenergy $\delta^2\mathscr{E} + \delta^2\mathscr{C}_0$ is negative definite. This is not possible with the PEM (e.g. the shallow-water equations) because in this case the potential vorticity field does not determine uniquely the velocity and mass fields: the null space in the derivation of $(\delta\boldsymbol{u}, \delta\zeta)$ from $\delta\boldsymbol{q}$ are the Poincaré waves, which are explicitly filtered out in the quasi-geostrophic approximation (Ripa 1981*b*).

Arnol'd's second theorem is usually derived assuming $\delta\gamma \equiv 0$. Certainly, constancy of Kelvin circulations (2.14) implies that any $\delta\gamma$ is time-independent, and a perturbation $(\delta q, \delta\gamma)$ from (Q, Γ) is equivalent to the perturbation $(\delta q, 0)$ from $(Q, \Gamma + \delta\gamma)$. However, it ought to be proved that going from (Q, Γ) to (q, γ) in two steps, passing through $(Q, \Gamma + \delta\gamma)$, involves a well-defined norm; i.e. the stability theorem with $\delta\gamma \not\equiv 0$ is unavoidable. This is done next:

Let the perturbation streamfunction be split into the parts 'induced' by δq and $\delta\gamma$, the latter chosen to have vanishing potential vorticity perturbation

$$\delta\psi_j(\boldsymbol{x}, t) = \delta\psi^q + \delta\psi^\gamma; \tag{3.7}$$

since $\delta\psi$ is a linear function of δq and $\delta\gamma$, albeit a non-local one, this decomposition is well posed. This is done by first performing the expansions

$$\delta q_j(\boldsymbol{x}, t) = \sum_{l,s} A_l^s(t)\, G_j^l\, \chi_s(\boldsymbol{x}), \tag{3.8a}$$

$$\delta\gamma_j^\nu - \delta\hat{\gamma}_j^\nu = \sum_l B_l^\nu G_j^l, \tag{3.8b}$$

where the G_j^l are the vertical normal modes (with eigenvalue μ_l), and the $\chi_s(\boldsymbol{x})$ are the eigensolutions of the Helmholtz equation (with eigenvalue λ_s^2), defined in (A 1), (A 2) and (A 4), respectively. In (3.8*b*), $\delta\hat{\gamma}$ represents the Kelvin circulation perturbation due to the expansion in (3.8*a*); this point will be clarified at the end of this section with the particular case of a channel.

The first contribution to (3.7) is then

$$\delta\psi_j^q(\boldsymbol{x}, t) = -\sum_{l,s} (\lambda_s^2 + R_l^{-2})^{-1} A_l^s(t)\, G_j^l\, \chi_s(\boldsymbol{x}). \tag{3.9}$$

The parameter R_l, equal to $(f_0^2\mu_l)^{-\frac{1}{2}}$, is the Rossby deformation radius corresponding to the lth vertical mode. The second contribution is

$$\delta\psi_j^\gamma(\boldsymbol{x}) = \sum_{\nu,l} C_l^\nu G_j^l \theta_l^\nu(\boldsymbol{x}), \tag{3.10}$$

where the C_l^ν are proportional to the B_l^ν, and the θ_l^ν, and the $\theta_l^\nu(\boldsymbol{x})$ are defined, in (A 5), in such a way that there is no potential vorticity perturbation associated to $\delta\psi^\gamma$.

Now, the wave energy, the integral of (3.2), takes the form $(\Delta-\delta)\mathscr{E} = [\delta\psi^\gamma, \delta\psi^\gamma]+2[\delta\psi^q, \delta\psi^\gamma]+[\delta\psi^q, \delta\psi^q]$; where $[\delta\psi^\gamma, \delta\psi^\gamma]$ is time independent, by construction. The key property (proved in Appendix A), that allows for the derivation of Arnol'd's second theorem even with $\delta\gamma \not\equiv 0$, is that the crossed term vanish identically, i.e.

$$2[\delta\psi^q, \delta\psi^\gamma] := \int_{\mathfrak{D}} \mathrm{d}^2x \left(\sum_{j=1}^{N} H_j \nabla\delta_j^q \cdot \nabla\delta\psi_j^\gamma + \sum_{j=a}^{\ell} g_j' \, \delta\zeta_j^q \, \delta\zeta_j^\gamma \right) \equiv 0. \tag{3.11}$$

Consequently, the only time-dependent term in the wave energy is $[\delta\psi^q, \delta\psi^q] \equiv \frac{1}{2}\sum_{l,s}|s_l^2|^2(\lambda_s^2+R_l^{-2})^{-1}\langle(G_j^l\chi_s)^2\rangle$. On the other hand, it is easy to prove that $\langle\delta q^2\rangle \equiv \sum_{l,s}|A_l^2|^2\langle(G_j^l\chi_s)^2\rangle$. From (3.1)–(3.4) it follows that if

$$\frac{\mathrm{d}\Psi_j}{\mathrm{d}Q_j} < -(\lambda_0^2+R_0^{-2})^{-1} \tag{3.12}$$

then the pseudoenergy $\delta^2\mathscr{E}+\delta^2\mathscr{C}_0$ – minus the trivial constant $[\delta\psi^\gamma, \delta\psi^\gamma]$ – is negative definite and therefore the basic flow is formally stable; λ_0^2 is the gravest eigenvalue of the Helmholtz equation (A 4) and R_0 is the deformation radius corresponding to the gravest vertical normal mode compatible with the boundary conditions. If a basic flow is unstable and the stability condition is violated by the λ and R of just a few modes, then a growing perturbation must have a finite amplitude in those modes.

Normed stability can be proved if there are two positive constants a and A, such that

$$(\lambda_0^2+R_0^{-2})^{-1} < a \leqslant -\sigma_j(Q_j, \delta q_j)/\delta q_j^2 \leqslant A \quad (\forall \delta q_j \neq 0); \tag{3.13}$$

this implies $[a-(\lambda_0^2+R_0^{-2})^{-1}]\langle\delta q^2\rangle \leqslant a\langle\delta q^2\rangle-(\Delta-\delta)\mathscr{E} \leqslant -2\Delta(\mathscr{E}+\mathscr{C}_0) \leqslant -2\Delta\mathscr{C}_0 \leqslant A\langle\delta q^2\rangle$. Consequently $\langle\delta q^2\rangle$, whose square root qualifies as a norm of the perturbation, is bounded, at any time, by its initial value times $A/[a-(\lambda_0^2+R_0^{-2})^{-1}]$.

The stability conditions (3.12) and (3.13) reduce to those of Arnol'd (1965, 1966) in the case of plane flow, $R_0^{-2} \equiv 0$; moreover, (3.12) becomes that of Benzi *et al.* (1982) for the 'equivalent barotropic' case in the infinite plane $\lambda_0^2 \equiv 0$. Since condition (3.12) is expressed in terms of the eigenvalues of the gravest modes, it does not change if the model is improved by adding more and more layers, all the way up to the continuously stratified case (with isopycnal horizontal boundaries) in which the conditions of Swaters (1986) and McIntyre & Shepherd (1987) are recovered. This property of the QGM is completely the opposite of the PEM, whose stability conditions change drastically with the vertical definition of the model (Ripa 1991*a*).

3.2. *Symmetric basic flow*

If the Hamiltonian has some spatial symmetry (or, equivalently, the corresponding momentum is conserved), $[\mathscr{M}, \mathscr{H}] = 0$, then solutions of the formal stability conditions (3.5) or (3.12) must be themselves invariant under the transformation generated by $\mathscr{M}$, in virtue of Andrews' theorem (1984; see also Ripa 1991*b*). If the basic flow is x-symmetric, then there is at most one Casimir $\mathscr{C}_1$ such that $\delta(\mathscr{M}^x+\mathscr{C}_1)=0$; if it is axisymmetric, then there is at most one Casimir $\mathscr{C}_2$ such that $\delta(\mathscr{M}^a+\mathscr{C}_2)=0$. In either case, one may be tempted to use the pseudomomentum $\mathscr{M}+\mathscr{C}$ in order to prove the stability of a basic flow that is symmetric but not necessarily steady; however such a state does not seem to exist for QGM.†

Consider, therefore, a basic flow which is both steady and symmetric: the optimum choice for $\mathscr{S}$ is an arbitrary combination of the pseudoenergy $(\mathscr{E}+\mathscr{C}_0)$ and the pseudomomentum $(\mathscr{M}^x+\mathscr{C}_1$ or $\mathscr{M}^a+\mathscr{C}_2)$, say

$$\mathscr{S}=\mathscr{E}-\alpha\mathscr{M}+\mathscr{C}, \tag{3.14}$$

with an arbitrary α, where $\mathscr{C}=\mathscr{C}_0-\alpha\mathscr{C}_1$ or $\mathscr{C}=\mathscr{C}_0-\alpha\mathscr{C}_2$, in the parallel or axisymmetric cases, respectively (α has dimensions of linear or angular velocity, in each case). Since $\delta\mathscr{M}^x/\delta q_j\equiv y$ and $\delta\mathscr{M}^a/\delta q_j\equiv-\frac{1}{2}(x^2+y^2)(=-\frac{1}{2}r^2)$, all the formulae for the steady basic flow carry on, with the sole replacement $\Psi_j(s)\rightarrow\Psi_j(s)+\alpha Y_j(s)$, in the parallel case, or $\Psi_j(s)\rightarrow\Psi_j(s)-\frac{1}{2}\alpha R_j(s)^2$, for the axisymmetric one, where $Y_j(Q_j)$ and $R_j(Q_j)$ are the inverse functions of $Q_j(y)$ and $Q_j(r)$, respectively.

In the parallel case, then, formal stability is guaranteed when there is any value of α, such that

$$\frac{U_j-\alpha}{Q_{j,y}}<0 \tag{3.15}$$

(first theorem), or

$$\frac{U_j-\alpha}{Q_{j,y}}>(\lambda_0^2+R_0^{-2})^{-1} \tag{3.16}$$

(second theorem), everywhere.‡ Furthermore, normed stability requires the existence of two constants b and B, such that

$$b\delta q^2\leqslant\int_{Q_j}^{Q_j+\delta q}[\Psi_j(s)-\Psi_j(Q_j)-\alpha Y(s)+\alpha Y(Q_j)]\,\mathrm{d}s\leqslant B\delta q^2, \tag{3.17}$$

and either $b>0$ (first theorem) or $B<-(\lambda_0^2+R_0^{-2})^{-1}$ (second theorem). Lipps (1963) and Pedlosky (1964) obtained normal modes stability conditions which are equivalent to the α term in (3.15), i.e. using pseudomomentum conservation, in the notation of this paper.

Instability conditions are that for any value of α both (3.15) and (3.16) must be violated somewhere; a growing perturbation must have an appreciable amplitude where that violation takes place, in order to be able to make $\Delta\mathscr{S}=0$.

In the following two sections, I will apply these results to two examples of quasi-geostrophic instability: the sinusoidal profile in a one-layer model with a free surface, and Phillips' (1954) two-layer problem with rigid horizontal boundaries; these models represent the two following steps from the plane flow studied by Arnol'd (1965, 1966). Optimum application of the second theorem, with both pseudoenergy and pseudomomentum conservation (the arbitrariness of α is crucial), results in nonlinear stability conditions, which happen to coincide with those for normal modes. Wave energy and Casimir integrals are calculated for that mode.

† Unlike the case of PEM (Ripa 1991*a*). This resembles the fact that a Rossby wave with vanishing zonal wavenumber (and thus symmetric) has also zero frequency (i.e. it is steady); this is not true for Poincaré waves.

‡ For axisymmetric basic flows, $(U-\alpha)/Q_y$ should be replaced by $-(\Omega-\alpha)\,r/Q_r$, where $\Omega_j(r)$ is the local angular velocity of the basic flow.

In both cases, the horizontal domain $\mathfrak{D}$ is chosen to be the channel $y_1 \leqslant y \leqslant y_2$, whose width is denoted by $L := y_2 - y_1$. The eigensolutions of the Helmholtz equation (A 4), used in the expansion of δq, are of the form

$$\chi_s(x) \propto \exp(\mathrm{i}k_s x) \sin(l_s(y-y_1)) \tag{3.18}$$

with $\sin(l_s L) = 0$ and $\lambda_s^2 = k_s^2 + l_s^2$; the gravest mode corresponds to $k_0 = 0$, $l_0 = \pi/L$, i.e. $\lambda_0 = \pi/L$. Notice that there is a finite $\delta\hat{\gamma}$ owing to those χ_s with $k_s \equiv 0$; this must be subtracted from the actual value of $\delta\gamma$ before doing the expansion (3.8*b*). The basis for $\delta\psi^{\gamma}$ is of the form $\theta_l^{\nu}(x) \propto \cosh((y-y_\nu)/R_l)$, $\nu = 1, 2$.

4. Sinusoidal profile in the equivalent barotropic model

Consider the system studied by Benzi *et al.* (1982): one-layer model with a finite deformation radius R; this corresponds to $N = 1$ in (2.1)–(2.3), and the choices (2.4*b*) and (2.5*a*), with $\tau_1 \equiv 0$. Consequently $q_1 = \nabla^2\psi_1 - f_0\zeta_0/H_1 + \beta y$ and $\zeta_0 = f_0\psi_1/g'$, i.e. (omitting the unnecessary subscript) $q \equiv \nabla^2\psi - R^{-2}\psi$ with $R^2 := g'H/f_0^2$. Formal stability conditions (3.15) and (3.16) for a parallel flow in this model read

$$\frac{U-\alpha}{\beta - U'' + R^{-2}U} < 0 \quad \text{or} \quad > \frac{1}{\lambda_0^2 + R^{-2}}; \tag{4.1 a, b}$$

the term $R^{-2}U$ on the left-hand side shows that only the problem without free surface ($R^{-2} = 0$) has the property of 'Doppler shift' (White 1982), which may be mistaken for a Galilean invariance.

Consider the zonal flow

$$U = U_0 \sin(\Lambda y) - \beta R^2; \tag{4.2}$$

the term βR^2 is chosen so that β effects cancel out in this problem. If the basic flow is unstable, then both conditions (4.1*a*) (first theorem) and (4.1*b*) (second theorem) must be violated, for any value of α. Violation of (4.1*a*) with $|\alpha|$ sufficiently large requires that $\beta - U'' + R^{-2}U$ have a zero within the channel. Violation of (4.1*b*) with $\alpha = -\beta R^2$ yields

$$\Lambda L > \pi, \tag{4.3}$$

as a necessary condition of instability; I will show next that this condition is also sufficient for normal modes instability. Making

$$\delta\psi^q = \epsilon\,\mathrm{Re}\,[F(y)\,\mathrm{e}^{\mathrm{i}k(x-ct)}] + O(\epsilon^2),$$

in the linearized potential vorticity equation, $(\partial_t - U\partial_x)\,\delta q + \delta v Q_y = 0$, it is easily obtained

$$(U-c)(F'' - (k^2 + R^{-2})F) - (\beta - U'' + R^{-2}U)F = 0, \tag{4.4}$$

and the boundary conditions $kF = 0$ at $y = y_1$ and $y = y_2$; this equation has the same properties as that obtained for the Rayleigh problem in the plane without rotation, $\beta = 0 = R^{-2}$ (Drazin & Howard 1966, pp. 7–9): The eigenvalue c at the onset of instability, i.e. for k^2 near the critical value k_c^2, is obtained looking for an eigenfunction of (4.4), belonging to the discrete set, with c equal to the value of U at the point where $\beta - U'' + R^{-2}U (= Q')$ vanishes: this is given by

$$F_c = \sin(l_c(y-y_1)), \quad c_c = -\beta R^2, \quad k_c^2 = \Lambda^2 - l_c^2, \tag{4.5}$$

where $l_c := \pi/L$. Therefore the sinusoidal profile is unstable to normal modes (k_c real) if (4.3) is satisfied, which is the same condition as that found in the non-rotating case

(Drazin & Howard, 1966, p. 35). The second theorem, with the optimum value of α, is in this case not only sufficient but also necessary for formal stability; proof of nonlinear stability is in this case a trivial matter, because for this basic flow $(\Delta-\delta)(\mathscr{C}_0-\alpha\mathscr{C}_1)\equiv\frac{1}{2}\delta^2(\mathscr{C}_0-\alpha\mathscr{C}_1)$, at $\alpha=-\beta R^2$, and therefore $(-\Delta\mathscr{S})^{\frac{1}{2}}$ can be used as a (time independent) norm of the perturbation; this corresponds to $b\equiv B$ in (3.17).

Even though the deformation radius R does not appear in the instability condition (4.3), its value is important for the dynamics, as shown next. Following Drazin & Howard (1966, pp. 12–13) integrate $F_c''(y)F(y)-F_c(y)F''(y)$ between $y=y_1$ and $y=y_2$, and then make $k^2\uparrow k_c^2$, to obtain

$$(k^2-k_c^2)\int_{y_1}^{y_2}\mathrm{d}yF_c^2\sim(c-c_c)\int_{y_1}^{y_2}\mathrm{d}y\left(\mathbb{P}\frac{1}{U-c_c}\mp\mathrm{i}\pi\delta(U-c_c)\right)F_c^2\frac{Q'}{U-c_c},\tag{4.6}$$

where $\mathbb{P}$ denotes the Cauchy principal value of the integral, and the minus (plus) sign corresponds to $\operatorname{Im}(c)$ positive (negative). In the case of the sine profile (4.2), using (4.5) in (4.6) and assuming $y_1=-y_2$, for simplicity,

$$2\pi(\Lambda+R^{-2})\,|\operatorname{Im}(c)|\sim\Lambda L|U_0|\,(k_c^2-k^2)\quad\text{as } k^2\uparrow k_c^2\tag{4.7}$$

is obtained; a finite deformation radius produces a weaker growth rate, for everything else fixed, than the case of the plane flow, $R^{-2}=0$.

Let me finish this section by presenting the value of the pseudoenergy and pseudomomentum integrals. At the onset of instability, $k^2=k_c^2-0$, the wave kinetic and potential energies and the wave Casimirs are found to be

$$\int\mathrm{d}^2x\left((\nabla\delta\psi^q)^2,R^{-2}(\delta\psi^q)^2,-\frac{U-\alpha}{\beta-U''+R^{-2}U}(\delta q)^2\right)=(\delta^2\mathscr{E}_k,\delta^2\mathscr{E}_p,\delta^2\mathscr{C}_0-\alpha\delta^2\mathscr{C}_1)$$
$$\propto(\Lambda^2,R^{-2},-\Lambda^2-R^{-2});$$

consequently, the wave pseudoenergy vanishes, $\delta^2(\mathscr{E}+\mathscr{C}_0)\equiv 0$, as it should for a growing or decaying perturbation. The integral of $\delta^2(\mathscr{C}_0-\alpha\mathscr{C}_1)$ is calculated, with the eigensolution in (4.5), for $\alpha=-\beta R^2$ so as to cancel the singularity at $y=0$. However, for $k^2<k_c^2$ it is $\delta^2\mathscr{C}_1\equiv 0$, because the wave pseudomomentum must also vanish, and $\delta^2\mathscr{M}\equiv 0$ for quasi-geostrophic models.

5. Phillips' problem

In this section, I will consider a two-layer system, with $\beta=0$, and with rigid and horizontal boundaries, i.e. $N=2$ in (2.1)–(2.3) and the choices (2.4*a*) and (2.5*a*), with $\tau_0\equiv\tau_2\equiv 0$. I will assume, furthermore, that the basic flow is parallel, with constant currents U_1 and U_2 in each layer; this is an example of *baroclinic* instability, whereas that of the last section was *barotropic*. This model was first developed by Phillips' (1951, 1954); Gill, Green & Simmons (1974) introduced the improvements of $\beta\neq 0$, topography, and unequal thicknesses (only the last one, $H_1\neq H_2$, is made here).

The potential vorticity for the basic state is given by $Q_j=y(U_j-U_{3-j})f_0^2/g'H_j$, $j=1,2$. Formal stability in the form of Arnol'd's first theorem (pseudoenergy positive definite) cannot be proved, because (3.15) is violated for any α; in order to overcome this difficulty Henrotay (1983) included dissipation and applied the Lyapunov method in the form $\mathrm{d}[\delta^2(\mathscr{E}-\alpha\mathscr{C}_1)]/\mathrm{d}t<0$, i.e. using only the sign-definite terms of pseudoenergy and pseudomomentum. Let me instead try for Arnol'd's second theorem.

The gravest mode corresponds to $\lambda_0 = \pi/L$, for the horizontal mode (3.18), and $\mu_0 = 0$ ($R_0^{-2} = 0$), for the vertical one (barotropic mode). Equation (3.16) then reads $(U-\alpha)/Q_y > L^2/\pi^2$, for some α and $j = 1, 2$, which implies

$$\frac{U_1}{U_1-U_2} - \frac{f_0^2 L^2}{g' H_1 \pi^2} > \frac{\alpha}{U_1-U_2} > \frac{U_2}{U_1-U_2} + \frac{f_0^2 L^2}{g' H_2 \pi^2}.$$

In order for an α satisfying these inequalities to exist, it is necessary that the left-hand side be larger than the right-hand side, namely

$$g' H_1 H_2/(H_1+H_2) > (f_0 L/\pi)^2 \tag{5.1}$$

(i.e. L/π must be smaller than the internal deformation radius). This is the stability condition derived from Arnol'd's second theorem, using the same bound for both coefficients $(U-\alpha)/Q_y$ in the wave Casimir; it is shown next that (5.1) is too restrictive as a stability condition (unless $H_1 = H_2$).

In order to prove Arnol'd's second theorem in its strongest form, let me start by making the expansion

$$\delta q_j(\boldsymbol{x}, t) = -\frac{f_0^2}{g' H_j} \operatorname{Re} \sum_s \xi_j^s(t)\, \chi_s(\boldsymbol{x}) \quad (j = 1, 2),$$

for a general potential vorticity perturbation. Doing likewise for $\delta\psi_j^q(\boldsymbol{x}, t)$ and using (2.1)–(2.3), it is found that

$$\delta\psi_1^q = \operatorname{Re} \sum_s \frac{(2\kappa_2+1)\,\xi_1 + \xi_2}{2(2\kappa_1\kappa_2 + \kappa_1 + \kappa_2)} \chi_s,$$

$$\delta\psi_2^q = \operatorname{Re} \sum_s \frac{\xi_1 + (2\kappa_2+1)\,\xi_2}{2(2\kappa_1\kappa_2 + \kappa_1 + \kappa_2)} \chi_s,$$

with

$$\kappa_j^s := \tfrac{1}{2}(k_s^2 + l_s^2)\, g' H_j / f_0^2. \tag{5.2}$$

(The superscript s in ξ and κ, and the arguments of ξ and χ, are omitted for simplicity). Using this in the energy part of the Lyapunov functional $\mathscr{S}$, (3.14), gives

$$(\Delta - \delta)\,\mathscr{E} \propto \sum_s \frac{(2\kappa_2+1)\,|\xi_1|^2 + (2\kappa_1+1)\,|\xi_2|^2 + 2\operatorname{Re}(\xi_1^* \xi_2)}{2(2\kappa_1\kappa_2 + \kappa_1 + \kappa_2)}; \tag{5.3}$$

the corresponding Casimir part is

$$(\Delta - \delta)\,\mathscr{C} \propto \sum_s \left(-\frac{U_1 - \alpha}{U_1 - U_2} |\xi_1|^2 + \frac{U_2 - \alpha}{U_1 - U_2} |\xi_2|^2 \right). \tag{5.4}$$

Notice that not only is the wave energy exactly quadratic, $(\Delta-\delta)\,\mathscr{E} \equiv \frac{1}{2}\delta^2\mathscr{E}$ (which is a peculiarity of the quasi-geostrophic models), but, in this problem, so is the wave Casimir, $(\Delta-\delta)\,\mathscr{C} \equiv \frac{1}{2}\delta^2\mathscr{C}$, because $(U-\alpha)/Q_y$ is constant in each layer. Therefore, $\Delta\mathscr{S}$ is exactly given by terms of the form $\sum \boldsymbol{M}_s^{ij} \xi_i^* \xi_j$, where each 2×2 matrix $\boldsymbol{M}_s$ is a function of λ_s^2.

The 'orthodox' application of the second theorem corresponds to finding a common bound to both terms in (5.4); instead, necessary and sufficient condition for $\sum \boldsymbol{M}_s^{ij} \xi_i^* \xi_j$ to be negative definite are $\operatorname{Tr}(\boldsymbol{M}_s) < 0$ and $\operatorname{Det}(\boldsymbol{M}_s) > 0$. This need only be required for the gravest mode (lowest λ_s^2), because the contribution of higher modes to $(\Delta-\delta)\,\mathscr{E}$ is smaller relative to their contribution to $(\Delta-\delta)\,\mathscr{C}$, i.e. the

	$\kappa \geqslant 1$ $\mathrm{Im}\,(c) \equiv 0$	$\kappa < 1$ $\mathrm{Im}\,(c) \neq 0$
$\delta^2\mathscr{E}_k$	$2\kappa^2$	2κ
$\delta^2\mathscr{E}_p$	$\kappa - 1$	$1-\kappa$
$\delta^2\mathscr{C}$	$(1+\kappa)\left(1-2\kappa^2 \mp 2\Xi\dfrac{U_1+U_2-2\alpha}{U_1-U_2}\right)$	$-(1+\kappa)$
$\delta^2\mathscr{S}$	$2(1+\kappa)\left(\kappa-\kappa^2 \mp 2\Xi\dfrac{U_1+U_2-2\alpha}{U_1-U_2}\right)$	0

TABLE 1. Contribution of a single normal mode to the wave kinetic & potential energies and to the wave Casimir; the sum of the three terms (pseudoenergy − α pseudomomentum) is an integral of motion.

derivative of each term in series (5.3) with respect to κ_1 or κ_2 is negative definite. The matrix $\boldsymbol{M}$, calculated from the sum of (5.3) and (5.4), gives

$$\mathrm{Tr}\,(\boldsymbol{M}) = \frac{1-2\kappa_1\kappa_2}{2\kappa_1\kappa_2+\kappa_1+\kappa_2},$$

$$\mathrm{Det}\,(\boldsymbol{M}) = \frac{\kappa_1\kappa_2(\kappa_1\kappa_2-1)}{(2\kappa_1\kappa_2+\kappa_1+\kappa_2)^2} - \frac{1}{4}\left[\frac{\kappa_1-\kappa_2}{2\kappa_1\kappa_2+\kappa_1+\kappa_2} - \frac{2\alpha-U_1-U_2}{U_1-U_2}\right]^2.$$

A sufficient condition for formal stability, $\mathrm{Tr}\,(\boldsymbol{M}) < 0$ and $\mathrm{Det}\,(\boldsymbol{M}) > 0$, is therefore obtained using that value of α that makes the expression between braces vanish: this condition is simply $\kappa_1\kappa_2 > 1$; using the values of κ_j (5.2) for the gravest mode, $k = 0$ and $1 = \pi/L$, it follows that

$$g'(H_1H_2)^{\frac{1}{2}} > 2(f_0L/\pi)^2 \tag{5.5}$$

is a sufficient condition for nonlinear, or normed, stability. This condition is more powerful than (5.1) (which corresponds to demanding $\kappa_1^{-1}+\kappa_2^{-1} < 2$, instead of $\kappa_1\kappa_2 > 1$), because it assures the stability of more systems. Notice that in order to obtain (5.1) and (5.5) it was crucial to be able to use an arbitrary α.

I will now calculate the normal modes (further details are given in Appendix B). Using

$$\psi_j = -U_jy + \epsilon\,\mathrm{Re}\,[\hat{\psi}_j\,\mathrm{e}^{\mathrm{i}k(x-ct)}]\sin(ly) + O(\epsilon^2), \tag{5.6}$$

in the equations of motion, (2.12), results in the eigenvalue

$$c = \frac{\kappa_1(1+\kappa_2)\,U_1+\kappa_2(1+\kappa_1)\,U_2 \pm (U_1-U_2)\,\Xi}{\kappa_1+\kappa_2+2\kappa_1\kappa_2}, \tag{5.7}$$

and the eigenvector

$$\begin{pmatrix}\hat{\psi}_1\\ \hat{\psi}_2\end{pmatrix} = \begin{pmatrix}\kappa_2-\kappa_1\kappa_2 \mp \Xi\\ -\kappa_1+\kappa_1\kappa_2 \mp \Xi\end{pmatrix}, \tag{5.8}$$

where
$$\Xi^2 = \kappa_1\kappa_2(\kappa_2-1). \tag{5.9}$$

Short enough modes cannot grow, i.e. $(k^2+l^2)\,g'(H_1H_2)^{\frac{1}{2}} \geqslant 2f_0^2 \Rightarrow \mathrm{Im}\,(c) \equiv 0$, in agreement with (5.5); as pointed out above, nonlinear stability implies stability to normal modes disturbances.

Table 1 shows the kinetic and potential wave energy, $\delta^2\mathscr{E}_k$ and $\delta^2\mathscr{E}_p$, and the wave Casimir, $\delta^2\mathscr{C}$, all calculated to $O(\epsilon^2)$ and for the particular case $H_1 = H_2$, for simplicity (the expressions for the general case $H_1 \neq H_2$, are given in Appendix B). Notice that for just one growing, or decaying, normal mode, $\mathrm{Im}\,(kc) \neq 0$, it is $\delta^2\mathscr{S} \equiv 0$, because it is both $\delta^2\mathscr{S} =$ constant and $\delta^2\mathscr{S} \propto \exp[2\,\mathrm{Im}\,(kc)\,t]$; the 'lines' $\delta^2\mathscr{S} = 0$ in phase space (which represent a compensation between the wave energy, $\delta^2\mathscr{E}$, and the wave Casimir, $\delta^2\mathscr{C}_0 - \alpha\delta^2\mathscr{C}_1$) are the stable and unstable manifolds of linearized dynamics. Of course, a pair of growing and decaying normal modes, with opposite values of Im (kc), will have a finite contribution $\delta^2\mathscr{S} =$ constant $\neq 0$ (Held 1985), proportional to the product of both amplitudes, indicating that $(q, \gamma) = (Q, \Gamma)$ is a 'saddle point' in phase space.

In writing table 1 from equations (B 2)–(B 6), a common factor $\kappa^2|1-\kappa|$ was eliminated. For the more general case, $H_1 \neq H_2$, from analysis of expression (B 4), which gives $\delta^2\mathscr{S}$ for neutral modes $(\kappa_1\kappa_2 > 1)$, it is seen that $\delta^2\mathscr{S}$ might have either sign, but tends to zero as $\kappa_1\kappa_2 \uparrow 1$. For instance,

$$\delta^2\mathscr{S} = 2\kappa_1\kappa_2(\kappa_1\kappa_2-1)(\kappa_1+\kappa_2-\kappa_1^2-\kappa_2^2) \pm \Xi(\kappa_1-\kappa_2)[2\kappa_1\kappa_2(1-\kappa_1-\kappa_2)+\kappa_1+\kappa_2]$$

for the particular value $\alpha = \frac{1}{2}(U_1+U_2)$, and $\kappa_1\kappa_2 < 1$; as explained before, $\kappa_1\kappa_2 > 1 \Rightarrow \delta^2\mathscr{S} \equiv 0$. It is clear that for each of the stable perturbations, $\delta^2\mathscr{S} \rightarrow 0$ as the neutral stability curve is approached (i.e. $\kappa_1\kappa_2 \downarrow 1$).

Sakai (1989) studied the case $H_1 = H_2$ and pointed out that if one calculates eigensolutions for each layer, decoupled from the other, their pseudomomenta have opposite signs and their frequencies coincide inside the instability region of the coupled problem; he calls this a 'resonance' phenomenon. For the case of arbitrary depths this is found to occur, equating both expressions between braces in (B 1), at $\kappa_1\kappa_2 = \frac{1}{4}$, which is within the instability region $(\kappa_1\kappa_2 < 1)$. Sakai's method is clearly useful for the construction of a complete basis (for the expansion of the perturbation) and for *a priori* aiming to the instability region; however, I consider that there is an overuse of the word *resonance*, since both components are not always physically meaningful ones, but rather mathematical constructs. It is also interesting to point out that the pseudoenergies of the decoupled solutions need not have opposite signs.

6. Conclusions

The total variation of any functional $\mathscr{I}[\phi]$ may be considered a functional of the basic state and perturbation fields,

$$\Delta\mathscr{I} = \Delta\mathscr{I}[\Phi, \delta\phi];$$

that is also true for every term in the expansion of Δ $(\delta, \frac{1}{2}\delta^2$, etc.). For QGM, ϕ may be taken as the potential vorticity q, because the contribution of $\delta\gamma$ to the pseudoenergy and pseudomomentum integrals is orthogonal to that due to δq, as shown in Appendix A. If now $\delta\phi$ is expanded as

$$\delta\phi \sim \delta\phi^{(1)}\epsilon + \delta\phi^{(2)}\epsilon^2 + \delta\phi^{(3)}\epsilon^3 + \ldots$$

as $\epsilon \rightarrow 0$, and $\mathscr{I}$ is an integral of motion, $\Delta\dot{\mathscr{I}} = 0$, it follows

$$\delta\dot{\mathscr{I}}[\Phi, \delta\phi^{(1)}] = 0,$$

$$\delta\dot{\mathscr{I}}[\Phi, \delta\phi^{(2)}] + \tfrac{1}{2}\delta^2\dot{\mathscr{I}}[\Phi, \delta\phi^{(1)}] = 0,$$

etc. The first equation is usually a trivial one; e.g. for a parallel basic state, $\partial_x\Phi = 0$, and a non-symmetric perturbation, $\delta\phi \propto e^{ikx}$ with $k \neq 0$, $\delta\mathscr{I}$ vanishes identically. The second equation gives a balance between the rate of change of the 'wave' contribution to the integral of motion, $\frac{1}{2}\delta^2\mathscr{I}[\Phi, \delta\phi^{(1)}]$, and that due to the rectification of the 'mean flow', $\delta\mathscr{I}[\Phi, \delta\phi^{(2)}]$. In particular, from pseudoenergy conservation it is

$$\delta\dot{\mathscr{E}}[\Phi, \delta\mu^{(2)}] \equiv -\delta\dot{\mathscr{C}}_0[\Phi, \delta\mu^{(2)}] \equiv -\tfrac{1}{2}\delta^2\dot{\mathscr{E}}[\Phi, \delta\phi^{(1)}] \equiv \tfrac{1}{2}\delta^2\dot{\mathscr{C}}_0[\Phi, \delta\phi^{(1)}];$$

these equations may, of course, be proved directly from the evolution equations of $\delta\phi^{(1)}$ and $\delta\phi^{(2)}$ (and for a general, not necessarily parallel, basic flow). Finally, for the pseudomomentum (in the case of a symmetric basic flow) it is similarly found that

$$\delta\dot{\mathscr{M}}[\Phi, \delta\mu^{(2)}] \equiv -\delta\dot{\mathscr{C}}_1[\Phi, \delta\mu^{(2)}] \equiv -\tfrac{1}{2}\delta^2\dot{\mathscr{M}}[\Phi, \delta\phi^{(1)}] \equiv \tfrac{1}{2}\delta^2\dot{\mathscr{C}}_1[\Phi, \delta\phi^{(1)}].$$

For QGM it is $\delta^2\mathscr{M} \equiv 0$, and therefore the rate of change of the other three integrals vanish identically, and wave energy is positive definite. Consequently, the induced variation of mean flow energy is negative, whereas wave and mean flow momenta are both exactly zero; there is no net momentum induced by the waves. These are peculiarities of the QGM which, in general, are not found for those models based in the primitive equations (e.g. Kelvin–Helmholtz instability is an example where wave energy vanishes).

From the point of view of instability theory, quasi-geostrophic theory belongs (together with plane flow) to the class of models in which more information can be extracted from the integrals of motion, namely, conditions for nonlinear or normed stability and Arnol'd's second theorem. In contrast, primitive equations models at most can long for formal stability conditions in the sense of Arnol'd's first theorem, if layered in the vertical, or just normal modes conditions, if continuously stratified (Ripa 1990, 1991*a*).

The reason why it is possible here to pose the convexity estimates for the pseudoenergy integral, in order to prove normed stability, is that $(\Delta-\delta)\mathscr{E}$ is exactly quadratic in the perturbation, $(\Delta-\delta)\mathscr{E} \equiv \frac{1}{2}\delta^2\mathscr{E}$, whilst $(\Delta-\delta)\mathscr{C}_0$, which may have higher-order contributions ($\delta^n\mathscr{C}_0 \not\equiv 0$ for some n larger than two), is a functional of a single field, namely, δq. On the contrary, it is not possible to establish nonlinear stability for the shallow-water equations (or, in general, primitive equation layered models), because $\delta^3\mathscr{E}$ does not vanish, and furthermore is a functional of three independent fields, δu, δv and δh. Similar arguments follow for the pseudomomentum.

Probably the most striking difference between quasi-geostrophic and primitive equation models is that in the former it may be possible to prove stability in cases where $\delta^2\mathscr{C}_0$ (or $\delta^2\mathscr{C}_1$) is not positive definite (Arnol'd's second theorem), choosing a value of α such that $\delta^2\mathscr{C}_0-\alpha\delta^2\mathscr{C}_1$ is negative definite and larger in magnitude than $\delta^2\mathscr{E}-\alpha\delta^2\mathscr{M}$. This is done here for Phillips' baroclinic instability problem (without β or topographic effects) using, furthermore, the total variations of the integrals, not just the second-order one. It is thereby shown that

$$g'(H_1H_2)^{\frac{1}{2}} > 2(f_0L/\pi)^2$$

is indeed a condition for nonlinear quasi-geostrophic stability, for any shear. In contrast, the corresponding ageostrophic one ($f \equiv 0$), i.e. Kelvin–Helmholtz formal stability condition (Ripa 1990),

$$g'(H_1+H_2) > (U_1-U_2)^2,$$

does depend on the value of the shear, and has a completely different meaning: it guarantees that the wave energy be positive definite.

A similar analysis is done for a sinusoidal flow in the equivalent barotropic model: Arnol'd's second theorem gives

$$\Lambda L < \pi$$

(where Λ is the wavenumber of the basic flow), which again is found to be not only sufficient, but also necessary for stability. This condition happens to be the same as in the case without rotation: the main effect of a non-vanishing value of f_0 is a weakening of the growth rate. It is sometimes stated that the Coriolis force has a 'stabilizing' effect in planetary flows; it might be argued that this weakening of the growth rate is an example of such purported effect. However, in Phillips' problem, discussed above, the effect is the *opposite*: instability for $|f_0|$ above a certain threshold (this is also appreciated in the fully ageostrophic calculation of Sakai 1989). Coriolis effects may be 'stabilizing' or 'destabilizing' depending upon the example; there is no general principle.

Yet another way to analyse the differences between quasi-geostrophic and primitive equations models is to obtain, in both cases, the continuously stratified model as the limit of a layered case, as $N \to \infty$ and $H_j \to 0$. For primitive equations models, as the number of layers increases it is harder and harder to assure that the wave energy be positive definite (Ripa 1991*a*), to the point that there is no formal stability (let alone normed stability) condition in the continuously stratified case (Ripa 1990). For quasi-geostrophic models, on the other hand, wave energy is *a priori* positive definite, and there are no major difficulties with the continuously stratified limit. In fact, all the results of §§2 and 3 (Hamiltonian, Poisson bracket and the various stability conditions) are easily generalizable to the continuous case (McIntyre & Shepherd 1987), with isopycnal top and bottom boundaries, by simply replacing $\sum H_j \ldots$ for $\int \mathrm{d}z \ldots$ (non-isopycnal boundaries require the introduction of additional state variables; Holm 1986).

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Appendix A. Expansion basis for Arnol'd's second theorem

The vertical normal modes are defined by (Ripa 1986)

$$\left.\begin{aligned} G^l_{j+1} - G^l_j &= \mu_l g'_j F^l_j \quad (j = 1, \ldots, N-1), \\ G^l_j &= \frac{F^l_{j-1} - F^l_j}{H_j} (j = 1, \ldots, N), \end{aligned}\right\} \tag{A 1}$$

with the end conditions, corresponding to rigid or free boundaries, as in (2.4) and (2.5),

$$\left.\begin{aligned} \mu_l F^l_0 = 0 \quad &\text{or} \quad \mu_l g'_0 F^l_0 = G^l_1, \\ \mu_l F^l_N = 0 \quad &\text{or} \quad \mu_l g'_N F^l_N = G^l_N. \end{aligned}\right\} \tag{A 2}$$

From (A 1) it is easy to obtain

$$\sum_{j=1}^{N} H_j G^l_j G^m_j = F^m_0 G^l_1 - F^m_N G^l_N + \mu_l \sum_{j=1}^{N-1} g'_j F^l_j F^m_j.$$

Each of the first and second terms on the right-hand side either vanishes or is incorporated into the last summation, corresponding to the choice of a rigid (left) or free (right) boundary condition in (A 2). If both boundaries are rigid, left-most possibility in (A 2), then the gravest (barotropic) mode corresponds to $\mu \equiv 0$ and does not contribute to the potential energy. All other (baroclinic) modes have $\mu > 0$. For any choice of end conditions in (A 2), it is

$$\sum_{j=1}^{N} H_j G_j^l G_j^m \propto \delta_{lm}, \tag{A 3a}$$

$$\mu_l \mu_m \sum_{j=a}^{\ell} g'_j F_j^l F_j^m = \mu_l \sum_{j=1}^{N} H_j G_j^l G_j^m, \tag{A 3b}$$

where a and ℓ are defined in (2.4) and (2.5). There are N eigensolutions which form a complete and orthogonal basis for the expansion of δq and $\delta\gamma$, and thus of $\delta\psi$ (as done in (3.8)–(3.10)).

The $\chi(\boldsymbol{x})$, used in (3.8), are the eigensolutions of the Helmholtz equation

$$\left.\begin{aligned} (\nabla^2+\lambda_s^2)\,\chi_s(\boldsymbol{x}) &= 0 \quad (\boldsymbol{x}\in\mathfrak{D}),\\ \chi_s(\boldsymbol{x}) &= 0 \quad (\boldsymbol{x}\in\partial\mathfrak{D}),\end{aligned}\right\} \tag{A 4}$$

from which (3.9) follows. Finally, the $\theta(\boldsymbol{x})$, used in (3.10) are the solutions of

$$\left.\begin{aligned} (\nabla^2-R_l^{-2})\,\theta_l^\nu(\boldsymbol{x}) &= 0 \quad (\boldsymbol{x}\in\mathfrak{D})\\ \boldsymbol{n}\cdot\nabla\theta_l^\nu(\boldsymbol{x}) &= 1 \quad (\boldsymbol{x}\in\partial\mathfrak{D}_\nu),\\ \boldsymbol{n}\cdot\nabla\theta_l^\nu(\boldsymbol{x}) &= 0 \quad (\boldsymbol{x}\in\partial\mathfrak{D}_\iota;\, \iota \neq \nu),\end{aligned}\right\} \tag{A 5}$$

where $R_l^{-2} := \mu_l f_0^2$; the differential equation was chosen so that there is no potential vorticity perturbation associated with $\delta\psi^\gamma$.

The following orthogonality conditions between horizontal structure functions,

$$\int_{\mathfrak{D}} \mathrm{d}^2x \nabla\chi_r^* \cdot \nabla\chi_s = \lambda_s^2 \int_{\mathfrak{D}} \mathrm{d}^2x \chi_r^* \chi_s \propto \delta_{rs}, \tag{A 6a}$$

$$\int_{\mathfrak{D}} \mathrm{d}^2x \nabla\theta_l^\nu \cdot \nabla\chi_s^* = -f_0^2 \mu_l \int_{\mathfrak{D}} \mathrm{d}^2x \theta_l^\nu \chi_s^*, \tag{A 6b}$$

are easily obtained from (A 4) and (A 5).

The key ingredient to derive Arnol'd's second theorem with $\delta\gamma \neq 0$ is the orthogonality (3.11) of $\delta\psi^q$ and $\delta\psi^\gamma$, in the wave energy sense. In order to prove it, first of all notice that if $\delta\psi_j = \sum M_l G_j^l$ then $\delta\zeta_j = \sum f_0 \mu_l M_l F_j^l$, by virtue of definition (2.2) and (A 1). Then notice that in the vertical sums of (3.2), crossed terms corresponding to different vertical modes vanish identically, because of the orthogonality conditions (A 3). Therefore, the horizontal integral of (3.11) reduces, after using (A 3b), to that of $\nabla\theta_l^\nu \cdot \nabla\chi_s^* + f_0^2 \mu_l \theta_l^\nu \chi_s^*$, which vanishes because of (A 6b).

Appendix B. Normal modes of Phillips' problem

Substitution of (5.6) in (2.12) results in the $O(\epsilon)$ linearized equations

$$(U-c)\,\hat{q}+Q_y\,\hat{\psi} = 0, \text{ in each layer, i.e. the two equations}$$
$$[2\kappa_j(U_j-c)+U_{3-j}-c]\,\hat{\psi}_j-(U_j-c)\,\hat{\psi}_{3-j} = 0, \tag{B 1}$$

where the κ are given in (5.2). Requiring the determinant of this homogeneous system to vanish, the eigenvalue is obtained (5.7). The eigenvector is of the form

$$\hat{\psi}_1 \propto C_1[2\kappa_2(U_2-c)+U_1-c]+C_2(U_1-c),$$
$$\hat{\psi}_2 \propto C_2[2\kappa_1(U_1-c)+U_2-c]+C_1(U_2-c),$$

where the C_j are arbitrary; choosing them as $C_j \propto \kappa_{3-j}$ results in the expression (5.8). Calculating the wave energies and Casimir for just one normal mode results in the following:

For $\kappa_1 \kappa_2 > 1$ (neutral modes) it is

$$\delta^2 \mathscr{E}_{\mathrm{k}} = 2\kappa_1 \kappa_2 [\kappa_1 \kappa_2 (\kappa_1 + \kappa_2 - 2) \pm \Xi(\kappa_1 - \kappa_2)], \tag{B 2a}$$

$$\delta^2 \mathscr{E}_{\mathrm{p}} = \tfrac{1}{2}(\kappa_1 + \kappa_2 - 2\kappa_1 \kappa_2)^2, \tag{B 2b}$$

for the kinetic and potential wave energies, and

$$\delta^2 \mathscr{C} = (2\kappa_1 \kappa_2 - 1 \pm 2\Xi (U_1 + U_2 - 2\alpha)/(U_1 - U_2))\,\Theta \tag{B 3}$$

for the wave Casimir, where $\Theta = \kappa_1 \kappa_2 - (\kappa_1 \kappa_2 - \frac{1}{2})(\kappa_1^2 + \kappa_2^2) \mp \Xi(\kappa_1^2 - \kappa_2^2)$, and Ξ is given by (5.9). Equation (B 3) with $\alpha = 0$ gives $\delta^2 \mathscr{C}_0$, i.e. the Casimir contribution to the pseudoenergy; the term proportional to α, on the other hand, gives $\delta^2 \mathscr{C}_1$, the Casimir contribution to the pseudomomentum. Finally, adding (B 2) and (B 3), the wave contribution to the general second-order integral of motion 'pseudoenergy $-\,\alpha$ pseudomomentum' is obtained

$$\begin{aligned}\delta^2 \mathscr{S} = {} & 2\kappa_1 \kappa_2 (\kappa_1 \kappa_2 - 1)(\kappa_1 + \kappa_2 - \kappa_1^2 - \kappa_2^2) \\ & \pm \Xi((\kappa_1 - \kappa_2)(2\kappa_1 \kappa_2 (1 - \kappa_1 - \kappa_2) + \kappa_1 + \kappa_2) + 2(U_1 + U_2 - 2\alpha)/(U_1 - U_2)\,\Theta).\end{aligned} \tag{B 4}$$

The corresponding expressions for just one growing, or decaying, normal mode ($\kappa_1 \kappa_2 \leqslant 1$) are

$$\delta^2 \mathscr{E}_k = 2\kappa_1 \kappa_2 (\kappa_1 + \kappa_2 - 2\kappa_1 \kappa_2), \tag{B 5}$$

for the wave kinetic energy, (B 2*b*) for the potential one, and

$$\delta^2 \mathscr{C} = 2\kappa_1^2 \kappa_2^2 - \tfrac{1}{2}(\kappa_1 + \kappa_2)^2. \tag{B 6}$$

Addition of (B 2*b*), (B 5) and (B 6) gives $\delta^2 \mathscr{S} \equiv 0$, as it should, because it is both time independent and proportional to $\exp[2\,\mathrm{Im}\,(kc)\,t]$.

The expressions in table 1, are those in this Appendix for the particular case $\kappa_1 = \kappa_2$, without the common factor $\kappa^2|1-\kappa|$.

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